Equations for
Capacitor Measurement
using an Auto-balancing Bridge

Dr. Timothy E. Stinchcombe†

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†The author is employed as a Test Development Engineer with Heber Ltd., Stroud, UK. (www.heber.co.uk)
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1 Introduction

In the world of in-circuit test it is fairly easy to pick up the basics of how resistors (and for that matter, jumpers, diodes and transistors) are tested, using the standard inverting op amp set-up (given the grand title of ‘auto-balancing bridge’), and how to apply ‘guard points’ in order to solve particular problems of testing these devices in-circuit. However, move beyond this simple scenario, to the testing of reactive devices, i.e. capacitors and inductors, and you quickly discover there is a paucity of decent, detailed information readily available. I feel this lack of information puts me at a greater disadvantage when trying to understand and resolve testing issues involving these devices.

In the course of some discussion on capacitance measurement at the newly-established in-circuit test forum on Agilent’s website, [2], a small extract from an old manual was posted by one of the Agilent moderators, [3], and this contained the detail that I felt I was missing (at least for this particular testing configuration). It is the intention of this note to expand on that detail, and to hopefully present it in a more accessible way.

My primary interest is how capacitors are tested, particularly on the Agilent HP3070 platform: it must be stressed that since Agilent do not publish the details of their hardware, there can be no guarantees that what is presented here is actually applicable to the 3070! What follows is laced with presumption and ‘it is probably something like this’!

2 The Auto-balancing Bridge Method

2.1 The basic set-up

The basic set-up for the auto-balancing bridge is (see [1] for example):

![Auto-balancing Bridge Diagram]

where: ‘A’ is a high-gain amplifier (the ‘measuring operational amplifier’ in the HP3070 case); \( R_{ref} \) is a known, precision resistor; and \( Z_X \) is the unknown impedance (a capacitor...
in the case of particular interest), with real and imaginary components \( X_R \) and \( X_I \), that is:

\[
Z_X = X_R + jX_I.
\]

[I should point out here that there is a great deal of supposition on my part that this is how the HP3070 works—reference [1] goes on to show the much more complicated arrangement used by the Agilent 4294A Impedance Analyzer, utilizing null detectors, phase detectors, integrators etc., but from what I know of the 3070, gleaned mainly from the block diagrams in the service manuals, I do not believe it to have anywhere near the same level of complexity.]

Let the amplifier output voltage be \( V_{out} = V_{outn} + jV_{outi} \), and (because of the negative-feedback effect of the high-gain amplifier), we can equate the currents in \( Z_X \) and \( R_{ref} \):

\[
\frac{V_{in}}{X_R + jX_I} = -\frac{V_{outn} + jV_{outi}}{R_{ref}},
\]

which on re-arranging gives

\[
X_R + jX_I = -\frac{V_{in}R_{ref}}{V_{outn} + jV_{outi}}.
\]  

(1)

On taking absolute values

\[
|X_R + jX_I| = |Z_X| = \frac{V_{in}R_{ref}}{|V_{outn} + jV_{outi}|}
\]

we get an expression for the magnitude of \( Z_X \) in terms of the magnitude of the output voltage and the other known quantities

\[
|Z_X| = \frac{V_{in}R_{ref}}{V_{out}}.
\]  

(2)

[As a quick check, if \( Z_X \) is merely a resistor, \( R_X \), then this is simply

\[
R_X = \frac{V_{in}R_{ref}}{V_{out}}
\]

as we would expect from the simple inverting op amp set-up (albeit with the loss of sign due to taking absolute values).]

So much for the magnitudes: now let’s look at the phases. Take the phase of the voltage at the amplifier output to be

\[
\theta = \arg V_{out} = \tan^{-1} \frac{V_{outi}}{V_{outn}}
\]

From expression (1), if we clear the complex operator \( j \) from the denominator on the right-hand side, and equate real and imaginary parts on both sides, we get

\[
X_R = -kV_{outn} \quad \text{and} \quad X_I = kV_{outi},
\]
where \( k = \frac{V_{in} R_{ref}}{(V_{out}^2 + V_{out}^2)} \) is real-valued. Then
\[
\theta_Z = \arg Z_X = \tan^{-1} \frac{V_{out}}{-V_{out}} = -\tan^{-1} \frac{V_{out}}{V_{out}} = -\arg V_{out} = -\theta,
\]
i.e. the argument of the complex impedance \( Z_X \) is directly related to the phase of the amplifier output. Now
\[
X_R = |Z_X| \cos \theta_Z, \quad \text{and}
\]
\[
X_I = |Z_X| \sin \theta_Z
\]
from elementary complex number theory, and so if we can measure both the magnitude and phase of the amplifier output, \(|V_{out}|\) and \(\theta\), then from (2) and (3) we can determine the real and imaginary components of the impedance \( Z_X \). In particular, from (4), we can determine the reactive component of \( Z_X \):
\[
X_i = |Z_X| \sin \theta_Z
\]
\[
= \frac{V_{in} R_{ref}}{|V_{out}|} \sin(-\theta),
\]
and so
\[
X_i = -\frac{V_{in} R_{ref}}{|V_{out}|} \sin \theta.
\]
Thus for the particular case of \( Z_X \) being mainly a capacitive impedance, depending on how its reactive component \( X_i \) is made up, we may be able to work backwards from this expression to extract the actual value of the capacitance. What form \( X_i \) takes depends on just how we decide to model the capacitor: amongst the many models in common use (for example, reference [1] discusses how real components depart from their ideal counterparts), we will look at two simple ones, in turn, in the following two sections.

### 2.2 Using a simple series representation of the capacitor

Let’s assume that our capacitor behaves reasonably like a series combination of (small) resistor plus an ideal capacitor:

![Series Representation of Capacitor](image)

(Apparently this is good for smaller-valued capacitors.) The impedance for this is immediately
\[
Z_X = R_s + \frac{1}{j\omega C_s} = R_s + jX_s,
\]
on putting \( X_s = -1/(\omega C_s) \). Here \( X_s \) is equivalent to \( X_i \) of section 2.1 above, so substituting \( X_s \) into expression (5) gives
\[
-\frac{1}{\omega C_s} = X_s = X_i = -\frac{V_{in} R_{ref}}{|V_{out}|} \sin \theta,
\]
and simple re-arrangement gives the value of the capacitance in terms of all the other knowns

\[ C_s = \frac{|V_{out}|}{\omega V_{in} R_{ref}} = \frac{|V_{out}|}{2\pi f V_{in} R_{ref} \sin \theta}. \]  

(6)

The above assumes that the phase and magnitude of the amplifier output is measured—if instead these are resolved into their real and imaginary components, and noting that

\[ |V_{out}| = \sqrt{V_{out}^2 + V_{out}^2} \]

and

\[ \sin \theta = \frac{V_{out}}{\sqrt{V_{out}^2 + V_{out}^2}} \]

then the above expression becomes

\[ C_s = \frac{\sqrt{V_{out}^2 + V_{out}^2 \sqrt{V_{out}^2 + V_{out}^2}}}{2\pi f V_{in} R_{ref} V_{out}} = \frac{V_{out}^2 + V_{out}^2}{2\pi f V_{in} R_{ref} V_{out}}. \]

This is in agreement with the equivalent expression in reference [3].

### 2.3 Using a simple parallel representation of the capacitor

The other simple case is to take the capacitor as being like a (large) resistor in parallel with an ideal capacitor:

(So here the resistor reflects the leakage often seen in larger capacitors.) Deriving the impedance is not as straightforward as the series case—starting from

\[ \frac{1}{Z_X} = \frac{1}{R_p} + \frac{1}{jX_p}, \]

where \( X_p = -1/(\omega C_p) \), re-arranging gives a less-friendly expression:

\[ Z_X = \frac{j R_p X_p}{R_p + j X_p} = \frac{j R_p X_p (R_p - j X_p)}{R_p^2 + X_p^2} = \frac{R_p X_p}{R_p^2 + X_p^2} (X_p + j R_p). \]

(7)

On equating the imaginary part of this to the \( X_1 \) of section 2.1 above,

\[ X_1 \equiv \frac{R_p^2 X_p}{R_p^2 + X_p^2} \]

we see we need to rid the \( R_p \) terms in order to be able to get at \( X_p \). However, from (7) we have

\[ |Z_X| = \frac{R_p X_p}{R_p^2 + X_p^2} \sqrt{X_p^2 + R_p^2} = \frac{R_p X_p}{\sqrt{R_p^2 + X_p^2}} \]
and then we can form
\[ \sin \theta_Z = \frac{\text{Im}(Z_X)}{|Z_X|} = \frac{R_p^2 X_p}{R_p^2 + X_p^2} \cdot \frac{R_p X_p}{\sqrt{R_p^2 + X_p^2}} = \frac{R_p}{\sqrt{R_p^2 + X_p^2}}, \]

so now
\[ X_p \equiv \frac{R_p^2 X_p}{R_p^2 + X_p^2} = X_p \sin^2 \theta_Z. \]

Substitute this into (5):
\[ X_p \sin^2 \theta_Z = -\frac{V_{in} R_{ref}}{|V_{out}|} \sin \theta, \]
then
\[ -\frac{1}{\omega C_p} \sin^2 \theta = -\frac{V_{in} R_{ref}}{|V_{out}|} \sin \theta \]

as \( X_p = -1/(\omega C_p) \) and \( \theta_Z = -\theta \) (from (3)), which can now be re-arranged easily to extract the capacitance
\[ C_p = \frac{|V_{out}| \sin \theta}{\omega V_{in} R_{ref}} = \frac{|V_{out}| \sin \theta}{2\pi f V_{in} R_{ref}}. \quad (8) \]

Substituting for \( |V_{out}| \) and \( \sin \theta \) as before, gives the result in terms of the resolved components
\[ C_p = \frac{\sqrt{V_{out}^2 + V_{out}^2}}{2\pi f V_{in} R_{ref}} \frac{V_{out}}{\sqrt{V_{out}^2 + V_{out}^2}} = \frac{V_{out}}{2\pi f V_{in} R_{ref}}, \]
which again is the same as the result in [3].

3 Conclusion

Two expressions have been derived which show how a capacitor might be measured using an auto-balancing bridge, when assuming either a simple series model, equation (6), or a parallel one, (8), and where either the phase and magnitude, or real and imaginary components, of the bridge output voltage are measured.

References

